37 (MAT–1) 1·5

2023

(held in 2024)

MATHEMATICS

Paper : 1·5

(Tensor and Mechanics)

Full Marks : 80

Time : Three hours

The figures in the margin indicate full marks for the questions.

PART–A

(Objective-type questions)

Marks : 32

1. Answer ***all*** questions :

   (a) Prove that $\delta^p_q \delta^q_r = \delta^p_r$. 2

   (b) Prove that A^k_k is a scalar. 2

(c) For the metric :

$ds^2 = 5dx^2 - 6dxdy + 3dy^2$, find g. 2

(d) Prove that $\Gamma ij, k = \Gamma ji, k$. 2

(e) In case of forces in three dimensions, the intensity $\vec{R}$ of the wrench, Poinsot's moment k and the pitch p of the wrench are connected by the relation

(i) $p = \frac{K}{R}$ (ii) $p = \frac{R}{K}$ (iii) $p = RK$

(iv) $p = \sqrt{RK}$

Choose the correct answer. 1

(f) Forces $(X,0,0)$, $(0,Y,0)$, $(0,0,Z)$ act at the points $(0,0,c)$, $(a,0,0)$, $(0,b,0)$. Show that the pitch p of the wrench is given by

$$p = \frac{aYZ + bZX + cXY}{X^2 + Y^2 + Z^2}$$ 4

(g) A system of forces acting on a rigid body can be reduced to a single resultant force $\vec{R} = 2\hat{i} - 2\hat{j} + \hat{k}$ acting at 0 together with a couple of moment $\vec{G} = 2\hat{i} - 3\hat{j} + 6\hat{k}$. Write down the direction cosines of the central axis. 1

(h) The angular momentum $\vec{R}$ of a particle is given by

$$\vec{R} = 6t^2\hat{i} - (2t+1)\hat{j} + (12t^3 - 8t^2)\hat{k}.$$

Find the torque at time $t = 0$. 2

(i) A particle of mass m moves along the x-axis under the influence of a conservative force field having potential $V(x)$. If the particle is located at the positions x_1 and x_2 at respective times t_1 and t_2, prove that if E is the total energy,

$$t_2 - t_1 = \sqrt{\frac{m}{2}} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - V(x)}}$$ 4

(j) A particle describes the cardioid $r = a(1 - \cos\theta)$ under a force P directed towards the pole. Show that the law of force is $P \propto \frac{1}{r^4}$. 4

(k) A particle moves on a smooth sphere under no forces except the pressure of the surface; show that its path is given by the equation $\cot\theta = \cot\beta \cos\phi$, where θ and ϕ are its angular coordinates. 4

(l) Show that the moment of inertia of an elliptic area of mass M and semi-axes a and b about a diameter of length $2r$ is

$$\frac{1}{4} M \frac{a^2 b^2}{r^2}.$$ 4

PART–B

(Subjective-type questions)

Marks : 48

2. Answer *(a)* **or** *(b)* : 6

(a) Show that every second order tensor covariant or contravariant can be expressed as the sum of a symmetric tensor and a skew-symmetric tensor.

(b) Prove that g_{ij} is a covariant symmetric tensor of rank 2.

3. Answer *(a)* **or** *(b)* : 6

(a) If $A_{ij} = B_{i,j} - B_{j,i}$, prove that $A_{ij,k} + A_{jk,i} + A_{ki,j} = 0$

(b) Show that a vector of constant magnitude is orthogonal to its intrinsic derivative in any direction.

4. Answer *(a)* **or** *(b)* : 6

(a) .A single force $\vec{R}$ is equivalent to a force (X, Y, Z) acting at the origin together with the couple of moment $\vec{G} = (L, M, N)$. Show that $|\vec{R}| = \sqrt{X^2 + Y^2 + Z^2}$ and the equation of its line of action is

$$\frac{yZ - zY}{L} = \frac{zX - xZ}{M} = \frac{xY - yX}{N} = 1$$

(b) Two forces act, one along the line $y = 0$, $z = 0$ and the line $x = 0, z = c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is $(x^2 + y^2)z = cy^2$.

5. Answer *(a)* **or** *(b)* : 6

(a) A regular hexagon *ABCDEF* consists of six equal rods which are each of weight *W* and freely jointed together. The hexagon rests in a vertical plane and *AB* is in contact with a horizontal table *j* if *C* and *F* be connected by a light string, prove that is tensor is $W\sqrt{3}$.

(b) A stiff horizontal beam AB is supported at one end A and at some other point C. If the greatest possible uniformly distributed weight is to be placed upon it without breaking it, show that C must divide the beam in the ratio $\sqrt{2}+1:1$.

6. Answer *(a)* **or** *(b)* : 6

(a) Two particles of masses m_1 and m_2 move so that their relative velocity is $\vec{V}$ and the velocity of the centre of mass is $\vec{V}_0$. Show that the total kinetic energy of the system is given by $T = \frac{1}{2}M\vec{V}_0^2 + \frac{1}{2}\mu\vec{V}^2$,

where $M = m_1 + m_2$ and $\mu = \frac{m_1 m_2}{m_1 + m_2}$.

(b) A force

$$\vec{F} = (x + 2y + az)\hat{i} + (bx - 3y - z)\hat{j} + (4x + cy + 2z)\hat{k}$$

acting on a paticle is conservative. Find the values of a, b, c.

7. Answer *(a)* **or** *(b)* : 6

(a) A particle is projected vertically upwards with a velocity U, in a medium, the resistance of which varies as the square of the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{2g} \log\left(1+\frac{U^2}{V^2}\right)$, where $V = \sqrt{\frac{g}{k}}$.

(b) A planet is describing an ellipse about the sun as focus; show that its velocity away form the sun is greatest when the radius vector to the planet is at right angles to the major axis of the path, and that it is then is $\frac{2\pi ae}{T\sqrt{1-e^2}}$; where $2a$ is the major axis, e the eccentricity, and T the periodic time.

8. Answer *(a)* **or** *(b)* : 6

(a) A particle is attached to one end of a string of length L, the other end of which is tied to a fixed point O. When the string is inclined at an acute angle α to the downward — drawn vertical the particle is projected horizontally and perpendicular to the string with a velocity V. Show that the tension of the string is given by

$$T = m\left\{\frac{V^2}{L} + g(3\cos\theta - 2\cos\alpha)\right\}$$

(b) Obtain the expressions for the components of the velocity and acceleration in (r, θ, z) system for a particle moving in space.

9. Answer *(a)* **or** *(b)* : 6

(a) At the vertex C of a triangle ABC, which is right angled at C, the principal axes are a perpendicular to the plane and two others inclined to the sides at an angle $\frac{1}{2}\tan^{-1}\frac{ab}{a^2-b^2}$.

(b) Two uniform rods, AB, AC are freely jointed at A and laid or a smooth horizontal table so that the angle BAC is a right angle. The rod AB is struck by a blow P at B in a direction perpendicular to AB; show that the initial velocity of A is $\frac{2P}{4m'+m}$, where m and m' are the passes of AB, AC respectively.
